Grading Aggregates

I—Mathematical Relations for Beds of Broken Solids of Maximum Density

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Equations for the grading of broken solids are set up to give beds of maximum density. A consideration of the voids in a bed of the coarsest particles leads to two types of solution: the one for intermittent grading and the other for continuous grading. Theoretically, the first should give the best packing, provided the proportions of practical application are developed. An example is given of the method of calculating the sizes to be used for the intermittent grading. The second solution gives a system which has probable advantages of workability and its grading can be more or less closely approximated using commercial materials. The solution of the first equation is made and given graphically and the results so obtained have been used in plotting curves for the minimum possible voids on mixing 2, 3, or 4 sizes of materials. The method of solution of the equation for continuous grading is indicated. No single curve for continuous grading is sufficient for all systems. The type of curve for maximum packing is determined by the voids in a bed of normally packed, sized material, and the ratio of the size limits of the system.

THE size composition of the system of aggregates which enter into the manufacture of mortar and concrete is a very important factor in determining the workability of the mix and the density of the product. A great deal has been done empirically to solve the problem of grading of aggregates, and the contributions of Feret (5), Fuller (3), Abrams (1), Graf (7), Talbot (8), and many others have been very helpful, but so far little success has been attained in the mathematical development of the laws of packing of beds of broken solids, which make up our mortar and concrete aggregates. In this article formulas are developed for two types of grading of aggregates. The first involves an intermittent grading, such as has been successfully worked out empirically by John J. Early of Washington, D. C., and the second is for the continuously graded materials most commonly used. The derivations have been made with the aid of certain assumptions and the proof of the equations must lie with experiment. This development has been carried out in close cooperation with Anderheg (2), whose data cover the experimental verification of these laws and their practical application. While the object in view during the course of this study was to set up equations which would be useful for mortar and concrete, it should be pointed out that the equations here set up might also be applicable to other systems when broken solids are involved. Among these might be asphaltic concrete, paint, putty, rubber, coal storage, fuel beds, catalytic masses, etc.

In a previous publication (6) the author has shown that in beds of broken solids containing two component sizes the composition of maximum density is one where the proportion of the larger size measured in absolute volume of particles is 1/(1 + V), where V is the fractional voids in a bed of sized material—that is, V is the volume of voids in a unit of total volume of bed. For practical purposes material may be considered "sized" if it passes one screen and stays on another which differs from it by the factor \( \sqrt{2} \); for instance, through 3 mesh, on 4. This relation holds if the percentages of voids are the same for both component sizes.

Systems of More Than Two-Component Sizes

The basic idea of the argument of the preceding paragraph is that if particles of infinitely small size are introduced into a bed of large pieces the small pieces will fill up the voids without changing the total volume. In practice it is found that introducing small pieces into the bed increases the total volume somewhat but the composition of maximum density is still given approximately by the proportion 1/(1 + V) until the sizes of the two components begin to approach each other. Extending the argument to systems of more than two-component sizes, assume that there are several sizes in the system and that each component size fills exactly the voids of the next preceding size, causing no increase in volume of the bed as a whole and leaving no excess material. This is a purely hypothetical case and is used only as a starting point. Such a system is roughly analogous to a telescope of many sections where each piece slides into the preceding one, or to the sets of hollow interfitting blocks of diminishing size which are used by children for building towers.

An equation will now be developed showing the relation between voids, size composition, and a number of component sizes for uniformly mixed beds of maximum possible density. Let Z be the proportion by absolute volume of the large size in a two-component bed of maximum density—that is, Z is the volume of large-sized material per unit volume of solid matter. Then

\[
Z = \frac{1}{1+V} \quad (1)
\]

Equation 1 holds only for the case where the solid particles of the different sizes have the same shape so that the voids in the bed of the sized material are the same for each constituent size. The more complicated case of varying voids will be taken up later. In Equation 1, V is the fraction of voids in a bed of sized material, whereas Z is the actual absolute volume occupied by the larger particles in a two-component system of maximum density, when the actual absolute volume of both large and small particles is unity. Call the diameter of the large particles \( d \). The amount of fine material of diameter \( d_1 \) which will exactly fill the interstices of \( d \) is 1 - Z, provided the small particles act as if they are infinitely small. Suppose that another set of infinitely small particles of diameter \( d_4 \) can be introduced into the voids of the second component. A better way may be to consider momentarily that each component is of finite size but that it has the packing properties of the infinitely small size.

If such an arrangement is assumed, the total absolute volume of each component size will be given by a series of terms of decreasing magnitude, the first term being Z and the second 1 - Z, as mentioned before. The numerical ratio
between the second and first term is \((1 - Z)/Z\). This same factor persists throughout, for the same ratio of voids is left within each constituent size as it is added. Thus the series becomes a geometric progression and

\[
\text{Total absolute volume of solids} = \frac{d_1}{Z} + \frac{d_2}{1 - Z} + \frac{d_3}{(1 - Z)^2} + \cdots = \frac{1}{1 + V} + \frac{1}{1 - Z} \left( \frac{1 - Z}{Z} \right)
\]

Since, according to Equation 1, \(Z\) equals \(1/(1 + V)\), the quantity \((1 - Z)/Z\), which is the ratio between terms, is equal to \(V\). Therefore, the equation may be written

\[
\text{Total absolute volume of solids} = \frac{d_1 + d_2 + d_3 + \cdots}{1 + V} = \frac{1}{1 + V} + \frac{1}{1 - Z} \left( \frac{1 - Z}{Z} \right)
\]

The number of terms in the numerator equals the number of component sizes in the system. This equation applies only when the voids in a bed of the sized material are the same for each component size. (The more general case will be discussed later.) Each term represents the absolute volume of the different component sizes in the mixture, the size being designated by the symbols \(d_1, d_2, \ldots\), above each term. If all the solid pieces have the same density, \(1/(1 + V)\) may be considered the actual absolute volume of all the solid pieces of the largest size, \(d_1\), in a system of maximum density of two or more components. Then the absolute volume of the particles of the second size, \(d_2\), will be \(V/(1 + V)\), of the third size, \(d_3\), \(1^2/(1 + V)\), etc., as was explained by Equation 2.

Equation 2 is for the hypothetical case where each size acts as if it were infinitely small. However, it was found experimentally \((?)\) that for two-component systems the equation was valid for systems where the ratio size (small to large) was as much as 0.2.

**Relations between Layered and Uniformly Mixed Systems**

If a system is made up of several different component sizes and if each size is placed on the bed as a separate layer, then the volume occupied by the entire bed will be

\[
G_T S
\]

where \(G_T\) = true specific gravity of particles in bed

\(G_s\) = apparent specific gravity of bed of sized material

\(S\) = sum of actual volumes of all particles of all constituent sizes, or summation of Equation 2 for as many terms or fractions of sized materials as are being considered.

Now suppose that all the layers of the different sizes of materials are so well mixed that each size of particle is uniformly distributed throughout the bed. As separate layers of sized materials are mixed there will be a decrease in volume. If the particles of the first component, \(d_1\), are quite large and all the other sizes, \(d_2, d_3, \ldots\), are very small (ideally, infinitely small), the voids of the large pieces will be exactly filled and the final volume of the bed will be only that originally occupied by the largest size, or

\[
\text{Volume} = \frac{G_T}{G_s} (S - Z)
\]

where \(Z = 1/(1 + V)\) as defined by Equation 1.

In this case, then, the decrease of volume upon mixing of sizes is

\[
\frac{G_T}{G_s} (S - Z)
\]

This is the ideal case. For actual systems the volume decrease will be less than the above quantity, or

\[
\text{Volume decrease} = y \frac{G_T}{G_s} (S - Z)
\]

where \(y\) is some factor ranging in value between 0 and 1.0. If the particles of the second component are infinitely small, \(y\) will be unity; if they are of the same size as the first component, \(y\) will be zero. (The value for intermediate size ratios will be discussed later.) The total volume of the mixed system, then, is

\[
\text{Volume} = \frac{G_T}{G_s} \left[ S - y(S - Z) \right]
\]

The weight of the entire mass of material is

\[
W = G_T S
\]

The density of any system is

\[
\text{Density} = \frac{\text{weight}}{\text{volume}} = \frac{G_T S}{G_s [S(1 - y) + yZ]}
\]

Hypothesize a system of broken solids with a certain limiting ratio between the smallest and largest size. The question of size composition for the greatest possible density then becomes, how many intermediate sizes shall there be between the smallest and the largest in order to obtain the maximum density? As more component sizes are added to the system the total volume of material, \(S\), increases but the value of the volume decrease factor, \(y\), diminishes, so that there are two variable tendencies and the maximum density will be found by striking a balance between the two; the problem thus becomes one of maxima and minima, and the key to the solution is found in Equation 7. Inspection of Equation 7 shows that for density to be a maximum, the term \([y(S - Z)]/S\) must be a maximum, considering the changing of values of the two variables \(y\) and \(S\) as the number of constituent sizes is changed.

It is now necessary to introduce a new variable, \(n\), which designates the number of component sizes added to the original single size of the system. It should be noted particularly that this value of \(n\) is one less than the total number of constituent sizes in the system. It is desired to determine when the quantity \(y(S - Z)/S\) is a maximum. Therefore, it is necessary to differentiate this quantity with respect to \(n\), set this derivative equal to zero, and solve this equation for \(n\). \(Z\) is a constant for a given system. Therefore,

\[
\frac{d}{dn} \left[ \frac{y(S - Z)}{S} \right] = \frac{y}{S} \frac{d(S - Z)}{dn} + \frac{S - Z}{S} \frac{dy}{dn}
\]

Performing the indicated differentiation and setting the derivative equal to zero,

\[
\frac{Z}{S(S - Z)} \frac{dS}{dn} = -\frac{dy}{dn}
\]

In order to solve for the composition of maximum density, the quantities \(S\) and \(y\) must be determined. The value of \(S\) is equal to the right-hand member of Equation 2. This equation is a geometrical progression, the first term being \(Z\) and the ratio between terms \((1 - Z)/Z\). As was pointed out above, this ratio equals \(V\). The sum of a geometrical progression is given by a formula of the general form:

\[
S = \frac{A(1 - R^n)}{1 - R}
\]
then by definition is independent of the size of particle except for relatively small containers or for particles so small that the absorbed air film is significant. Therefore, the ratio between the successive sizes of the two constituents. The percentage of voids in a bed of sized material, for a given amount of work of packing, is defined by Equation 13

\[ y = \frac{1}{2.62} \left( \frac{V}{V_0} \right)^{1/n} \]

\[ n \text{ is the number of component sizes added to the largest size in the system and } n+1 \text{ is the total number of sizes present.} \]

The derivative of Equation 11 is

\[ \frac{dS}{dn} = -\frac{V^{n+1} \ln V}{1 - V^n} \]

EVALUATING \( y \)—The factor \( y \), which appears in Equation 8, can be evaluated only experimentally; \( y \) is a number between zero and unity which represents the ratio of the decrease in total volume of the given materials, upon mixing, to the decrease if infinitely small particles were used. The larger the ratio of the small to the large particles the smaller \( y \) will become and, in the limit where the components become the same size, \( y \) will be equal to zero. It is understood that this discussion is limited to systems of maximum density.

Therefore, \( y \) varies as some function of the ratio between the two sizes in a two-component system or as the ratio of consecutive sizes in a system of many components.

Designating the diameter of the smallest particle in a given system as \( d_{n+1} \), that of the largest particle as \( d_1 \), and the ratio between the sizes of the smallest and largest particles as \( K \), then by definition

\[ K = \frac{d_{n+1}}{d_1} \]

Now the size composition for maximum density in a two-component system is dependent entirely upon the voids in sized beds of the two constituents. The percentage of voids in a bed of sized material, for a given amount of work of packing, is independent of the size of particle except for relatively small containers or for particles so small that the absorbed air film is significant. Therefore, the ratio between the successive sizes for maximum packing in one part of the system is the same as for any other part. In other words, for maximum packing in a system of many-component sizes, the ratio between the diameter of particles of successive sizes must be constant for the entire system. Therefore, in our ideal system

\[ \frac{d_1}{d_2} = \frac{d_2}{d_3} = \frac{d_{n+1}}{d_n} \]
system for maximum density, for several values of $V$. It should be noted that the ordinate is not $n$ but $n + 1$, for the first term of the series was dropped in our consideration of the maxima and the total number of sizes is therefore one more than the number of terms in the series considered.

It will be observed that the abscissas of Figure 2 are plotted on a logarithmic scale and that the size ratio decreases very rapidly. Thus for a system of maximum density to have three component sizes, the size ratio must be very small indeed.

Again, application of the correction to the size of the intermediate material is made by using the equation

$$\log_{10} \frac{d_1}{d_2} = \frac{1}{0.85} \log_{10} \frac{d_2}{d_3}$$

changing the dimension of the fine aggregate from 0.032 to 0.024 inch. The former is the mesh dimension of the 20-mesh sieve while the latter is that of the 28-mesh sieve. According to this calculation, the mix by volume would be close to 65 parts coarse, 25 of fine aggregate, and 10 of cement. This means that every particle would have to go into the place where it belonged and because of the harshness of such a mix some special method of placing, such as effective vibration, would be required.

Computed Values of Voids at Compositions of Maximum Density

The preceding discussion has been limited to the determination of compositions of minimum voids and no mention was made of the actual values of voids in the bed of the mixed materials. With the aid of the preceding equations it is a simple matter to compute these voids for compositions of maximum density. This has been done for systems in which the voids of the sized material are 30, 40, 50, and 60 per cent for 2, 2.5, 3, and 4 component sizes. The curves are shown in Figures 3 to 6. The curve for 2.5 component sizes was computed by means of the principles cited in the preceding example. The curves of Figures 3 to 6 should be useful in predicting the actual density of beds of a given size composition when intermittent grading for maximum density is used.

It may be assumed that the best combination of strength, durability, and economy can be secured with a concrete for which the aggregates are so chosen as to fill as much space as possible. For this purpose, according to the previous discussion, 3 component sizes would seem to offer the best practical solution. The cement may be regarded as one of these three sizes in view of the fact that the great bulk of the cement has no chance to hydrate before the concrete has been placed (3). Figure 2 shows that increasing the voids in a bed of the material increases the number of component sizes. Thus, for a system where the size ratio is 0.001 the number of component sizes for maximum density may be increased by increasing the voids in the coarse aggregate. This increase in voids may be accomplished by having the aggregate of more uniform size or less regular in shape. The conclusion is that to make a concrete of greatest possible density the coarse aggregate should be large, of uniform size, and of irregular shape.

If a system is such that 3 component sizes give maximum
density, then the proper size of the fine aggregate is given by the equation

$$d_3 = \sqrt{\frac{d_2 d_3}{d_3}}$$  \hspace{1cm} (20)$$

where $d_1 =$ average diameter of coarse aggregate

$d_2 =$ average diameter of fine aggregate

$d_3 =$ average diameter of cement  

This equation should apply as a first approximation even if the number of component sizes for maximum density, as shown by Figure 2, is not exactly 3.0.

Since the system is to have a symmetrical size distribution, the diameter of the sand particles should be the square root of the size ratio.

**Systems of Varying Voids and Densities**

The systems considered so far have been for uniform voids and density of solids. Equation 2 gives the proportions of absolute volumes of particle for the different sizes of such a system. By referring to the development of the relations for two component systems of maximum density (6) it will be seen that for the general case of varying voids and densities the proportions by weight are given by the equation

$$W_1 = 1 - W_1 + (1 - W_1) \left( \frac{1 - W_2}{W_1} \right)$$  \hspace{1cm} (21)$$

where

$$W_1 = \frac{(1 - V_1) S_1}{(1 - V_1) S_1 + V_2 (1 - V_2) S_2}$$  \hspace{1cm} (22)$$

$$W_2 = \frac{(1 - V_2) S_2 + V_1 (1 - V_1) S_1}{(1 - V_1) S_1 + V_2 (1 - V_2) S_2}$$  \hspace{1cm} (23)$$

where $V_1 =$ voids in coarse aggregate

$V_2 =$ voids in fine aggregate

$V_3 =$ voids in cement

$S_1 =$ true specific gravity of coarse aggregate

$S_2 =$ true specific gravity of fine aggregate

$S_3 =$ true specific gravity of cement

After the proportions by weight have been determined, the proportions by volume may be obtained by dividing the weight by the apparent specific gravity.

Apparent sp. gr. = (1 - voids) (true sp. gr.)

The case of varying densities may also be handled by computing on a volume basis first, using Equation 19; relative weights being determined by multiplying volume proportions by specific gravity.

**Determination of Mixture for Maximum Density**

A concrete is made of limestone averaging 2 inches in diameter, voids 48 per cent, specific gravity 2.5; fine sand, voids 42 per cent, specific gravity 2.65; cement, average size 0.001 inch, voids 52 per cent, specific gravity 3.10. What should be the average size of the sand particles and what should be the size composition by volume to produce a concrete of maximum density? What percentage of voids will there be in the bed after mixing?

Size ratio of system = $\frac{0.001}{2} = 0.0005$

From Figure 2 it can be seen that for a system of maximum density there should be approximately 3 component sizes.

**Continuous Grading of Sizes**

So far only those systems have been considered in which all of the constituents were carefully sized. This condition occurs only rarely in practice so it is necessary to consider the condition for maximum density when the bed of broken solids
is not of uniform size but is continuously graded between definite limits. The question is: What form of the grading curve will produce a bed of maximum density? The relations obtained should be particularly applicable to the choice of aggregates for concrete and cement products. The question of the relationship between workability and grading is of great importance, and it is suggested that an analogy must exist between workability and ball-bearing friction, which decreases as the diameter of the ball approaches that of the shaft. That is, particles of the next size smaller than the definite limits. The question is: What form of the grading designed according to the principles enunciated later have decreases as the diameter of the ball approaches that of the largest may be expected to act as ball bearings, and so on down the line of sizes. No attempt will be made at this time to carry the analogy beyond this qualitative statement. It might be said, however, that mixes made up with gradings designed according to the principles enunciated later have been found to be very workable.

It might be said, however, that mixes made up with gradings displays a maximum density for certain size compositions, it may be expected that the same qualitative rule will hold when the number of component sizes becomes very large and the grading becomes continuous. Presumably then, for each class of material there will be a certain constant ratio between the amounts of material of consecutive screen sizes which will give maximum packing for that system. Such a ratio will be designated by the symbol \( r \). Arbitrarily, \( r \) is the ratio of the amount of material through and on two consecutive screens to the amount on the next smaller screen when the screen sizes vary by the factor \( \sqrt{2} \). For example, the amount of material through 3 mesh and on 4 mesh shall be \( r \) times as much as that through 4 mesh on 6 mesh. It is understood that “amount” refers to absolute volume of particles. Measurement by weight is valid as long as true density is constant from size to size. It is recognized that the exact definition of \( r \) should be the ratio of the slope of the curve at points differing by one screen size but the idea of the quantity through and on consecutive sieves is exact enough for the purpose and easier to understand and apply. Obviously in a continuously graded system

\[
\text{Cumulative per cent} = \frac{S - S_i}{S_l - S_i}
\]

subscript \( s \) refers to smallest size and subscript \( l \) to largest. Screen-size divisions are proportional to \( \log d \). Substituting \( \log d \) for \( m \) in Equation 10 and using the appropriate subscripts and substitutions in the latter equation gives, on simplification,

\[
\text{Cumulative per cent} = \frac{r^{l+1} d_i - r^{l+1} d_i}{r^{l+1} d_i - r^{l+1} d_i} = \frac{r^{l+1} d_i}{r^{l+1} d_i} = 1
\]

where \( d_i \) = diameter of smallest particle

\( r \) = ratio between consecutive sizes where “consecutive” sizes may be conveniently defined

In a previous section of this paper it was shown that for intermittent grading, for maximum density, the quantity of small particles divided by the quantity of large particles was equal to \( V \), where \( V \) is the fractional voids in a bed of uniformly packed sized material. The definition of \( r \) given above reverses this ratio for convenience in plotting screen sizes, so as to give the curve for cumulative percentage of material passing the several sieves. Thus \( r \) is the quantity of large particles divided by the quantity of small particles. It is evident that if the quantities of large and small particles were comparable in both cases, \( r \) would be \( 1/V \). The quantities can be made comparable by a relatively simple device. \( 1/V \) stands for the ratio between sizes for maximum possible density in intermittent grading. In order to make the two systems similar it is necessary to set up the equation

\[
r = \frac{1}{V} \]

Therefore

\[
r = \frac{1}{V/m}
\]

where \( n \) = one less than number of component sizes for maximum density in intermittent grading (one less than ordinate of Figure 2)

\( m \) = number of screen-size intervals of ratio \( \sqrt{2} \) included between size limits of system, or one less than number of screens used

For example, suppose the ratio of diameters of large to small particles to be 128 and the voids in a normally packed bed of sized material to be 40 per cent. There are 15 screens in the entire system and \( m \) equals 14. According to Figure 2 there are 2.45 component sizes for maximum density. Thus \( n \) equals 1.45. Therefore

\[
r = \frac{1}{V/m} = \frac{1}{0.91} = 1.10
\]
30, 40, 50, and 60 per cent. The corresponding values of \( r \) are 1.12, 1.10, 1.075, and 1.058, respectively. It should be noted particularly that these values of \( r \) are only for ratios of amounts on screen sizes differing by the factor \( \sqrt{2} \). For other factors \( m \) will be different and hence \( r \) will be different.

The computed grading curves for these systems are given in Figure 7 plotted on the convenient logarithmic scale and in Figure 8 on the more awkward linear scale. It will be noted that according to the theory advanced in the preceding discussion the type of grading curve for maximum density is determined by two factors: (1) the percentage of voids in a normally packed bed of sized material, which depends on the shape of the particles and (2) the limiting size ratio of the system. Thus, no single grading curve is sufficient for all types of material but each must be determined by Equations 25 and 26.

The author feels that this theory as developed should be useful in the correlation of experimental data and in prediction of the proper mixes for greatest density. Such experimental data are discussed in the accompanying paper by F. O. Anderegg.

**Literature Cited**


### Grading Aggregates

**II—The Application of Mathematical Formulas to Mortars**

**F. O. Anderegg**

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According to the Furnas development, the arrangement of the amounts of the successive sizes depends upon the voids in the bed of the coarsest size aggregate present. A large number of voids means more fines to fill them. Two systems can be used to secure efficient packings, one having an intermittent or gap grading and the other a continuous grading. Examples of the former may be found in the work of John J. Early and in recent dam building by the Aluminum Company of America (7); the latter is approximated more or less closely in most stuccos, cast stones, masonry mortars, concretes, and even in the cement itself.

In the application of the continuous grading principle it is necessary to separate the system into convenient fractions, as by the use of a series of sieves, and to then define the relation between the amounts retained on the several sieves. This relation is covered by the value \( r \) in the Furnas Equation 25 and may be defined for practical purposes as the ratio of the amount retained on any one sieve of a series that retained on the next smaller. If the sieve dimensions had a constant ratio of ten, common logarithms could be conveniently used, but as most sieves have the ratio of two, or the square root of two, these values may be used as our logarithmic base. Table I gives a comparison of values for \( r \) for the several bases.

<table>
<thead>
<tr>
<th>Sieve Width (mm)</th>
<th>Logarithmic Basis</th>
<th>Value of ( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0</td>
<td>1.00</td>
<td>1.414</td>
</tr>
<tr>
<td>1 5</td>
<td>1.139</td>
<td>1.110</td>
</tr>
<tr>
<td>2</td>
<td>1.232</td>
<td>1.100</td>
</tr>
<tr>
<td>3</td>
<td>1.392</td>
<td>1.075</td>
</tr>
<tr>
<td>4</td>
<td>1.624</td>
<td>1.058</td>
</tr>
<tr>
<td>5</td>
<td>2.000</td>
<td>1.000</td>
</tr>
<tr>
<td>10</td>
<td>2.785</td>
<td>0.700</td>
</tr>
</tbody>
</table>

This means that if the sieves of 200-, 100-, 50-, 30-, etc., mesh sizes are used, the ratio of material passing a certain 1

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2 Contribution from the Portland Cement Fellowship, Mellon Institute of Industrial Research, Pittsburgh, Pa.

<table>
<thead>
<tr>
<th>Table I—Comparative Values of ( r ) in Furnas Equation for Logarithmic Base, or Ratio of Sieve-Opening Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sieve Width (mm)</td>
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<tr>
<td>------------------</td>
</tr>
<tr>
<td>1 0</td>
</tr>
<tr>
<td>1 5</td>
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<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
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<tr>
<td>10</td>
</tr>
</tbody>
</table>

### Packing of Shot

A series of lead shot was obtained, including buck shot of numbers 000, 00, 0, 1, 2, 3, and 4 BB shot, drop shot of numbers 1 to 12, and shot dust. Several shot of the different sizes were measured accurately and were not found to vary greatly from true spheres. The ratio of largest to smallest diameter was slightly greater than 8. The Furnas Equation 25 was solved for the five intermediate gradings of Table 11, and the amounts of the several sizes of shot were taken from the curves. While the grading of these shot was not quite continuous, the differences between sizes was small enough to give a close approach to continuity. A set of iron shot was obtained through the courtesy of the Pittsburgh Crushed Steel Company and treated in a similar manner. The ratio of largest to smallest was about 21. These shot were somewhat more irregular in shape than the lead shot. The results for dry rodding are given in Table II.

<table>
<thead>
<tr>
<th>Table II—Packing of Lead and Iron Shot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead Shot</td>
</tr>
<tr>
<td>% Packing</td>
</tr>
<tr>
<td>1.063</td>
</tr>
<tr>
<td>1.110</td>
</tr>
<tr>
<td>1.150</td>
</tr>
<tr>
<td>1.180</td>
</tr>
<tr>
<td>1.274</td>
</tr>
<tr>
<td>1.414</td>
</tr>
</tbody>
</table>

These packings could be reproduced within ±0.1 per cent, using the same container and same method of tamping (twenty-five times on each third with a small pointed rod). Because of the small amount of shot available, the container was small (180 ml.), and as a result a small boundary error was introduced. As this must be nearly the same for the